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Solutions to 10. Exercise sheet Algebraic Geometry I

Solution to Aufgabe 1 Let X be the topological space obtained as quotient of the disjoint union $\bigsqcup_{i\in I} X_i$ by the following equivalent relation: two elements $x, y \in \bigsqcup_{i\in I} X_i$ are equivalent if there exist indices $i, j \in I$ such that $x \in X_{i,j}$, $y \in X_{j,i}$ and $\varphi_{i,j}(x) = y$. Consider X endowed with the quotient topology. For every $i \in I$, let $\psi_i : X_i \to X$ be the natural inclusion. By construction, ψ_i is a homeomorphism onto an open subset of X for all $i \in I$, $\psi_i(X_{i,j}) = \psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{i,j}$ on $X_{i,j}$ for all $i, j \in I$, and the family $\{\psi_i(X_i)\}_{i\in I}$ is an open covering of X. Let U be an open subset of X, then $U_i := \psi_i^{-1}(U)$ is an open subset of X_i for all $i \in I$. Let $\mathcal{O}_X(U)$ be the set of elements $(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X_i}(U_i)$ such that $(\varphi_{i,j}^{\#})_{U_{j,i}}(s_j|_{U_{j,i}}) = s_i|_{U_{i,j}}$ for all $i, j \in I$. We note that $\mathcal{O}_X(U)$ has a natural structure of ring compatible with the ring structure of $\mathcal{O}_{X_i}(U_i)$ for all $i \in I$. The family of sets $\mathcal{O}_X(U)$ associated to the open subsets of X, together with the restriction maps induced by the restriction maps of the sheaves $\mathcal{O}_{X_i}$, gives a sheaf of rings on X that we denote by $\mathcal{O}_X$. Moreover, for every $i \in I$ and every open subset $V \subseteq X_i$, the inclusion $\mathcal{O}_{X_i}(V) \subseteq \mathcal{O}_X(\psi_i(V))$ is an isomorphism, and such isomorphisms are compatible with the restriction maps. Thus $\psi_i : X_i \to X$ is an isomorphism onto the open subscheme $\psi_i(X_i)$ of X for all $i \in I$, and $\psi_i = \psi_j \circ \varphi_{i,j}$ on $X_{i,j}$ for all $i, j \in I$.

Solution to Aufgabe 2 (a) From the lecture we know that

$$\begin{aligned}&\mathrm{Hom}_K(\operatorname{Spec} K, \mathbb{P}^n_K)\\ &\simeq \{(x,\varphi) : x \in \mathbb{P}^n_K, \varphi : k(x) \to K \text{ such that } \varphi(s) = s \in K, \forall s \in K \subseteq k(x)\}.\end{aligned}$$

Let $\{U_i := \mathbb{P}^n_K \setminus V(T_i)\}_{i=0,\dots,n}$ be the standard open affine covering of $\mathbb{P}^n_K$. Since $k(x) \cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ for every affine open subset $U = \operatorname{Spec} A \subseteq \mathbb{P}^n_K$ containing x and prime ideal $\mathfrak{p}$ of A corresponding to x, the inclusions $\mathrm{Hom}_K(\operatorname{Spec} K, U_i) \subseteq \mathrm{Hom}_K(\operatorname{Spec} K, \mathbb{P}^n_K)$ give a bijection

$$\mathrm{Hom}_K(\operatorname{Spec} K, \mathbb{P}^n_K) \simeq \bigcup_{i=0}^n \mathrm{Hom}_K(\operatorname{Spec} K, U_i).$$

Fix $i \in \{0,\dots,n\}$. Since $U_i \cong \operatorname{Spec} K[\frac{T_0}{T_i},\dots,\frac{T_n}{T_i}]$ is affine, there are bijections $\mathrm{Hom}_K(\operatorname{Spec} K, U_i) \simeq \mathrm{Hom}_{K-\text{algebras}}(K[\frac{T_0}{T_i},\dots,\frac{T_n}{T_i}], K) \simeq K^n$, where the last is given by the evaluation maps corresponding to the variables $\frac{T_0}{T_i},\dots,\frac{T_n}{T_i}$. We note that a morphism of K-algebras $\psi : K[\frac{T_0}{T_i},\dots,\frac{T_n}{T_i}] \to K$, given by $\psi(\frac{T_j}{T_i}) = x_j \in K$ for all $j \neq i$, has kernel the maximal ideal $(\frac{T_0}{T_i} - x_0,\dots,\frac{T_n}{T_i} - x_n)$, which is the maximal ideal corresponding

to the point $(x_0 : \cdots : x_n) \in \mathbb{P}^n(K)$ in $\mathbb{P}^n_K$, here we consider $x_i := 1$. Thus, $\mathrm{Hom}_K(\operatorname{Spec} K, U_i) \simeq \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(K) : x_i \neq 0\}$, and $\mathrm{Hom}_K(\operatorname{Spec} K, \mathbb{P}^n_K) \simeq \mathbb{P}^n(K) \simeq (K^{n+1} \setminus \{0\})/K^\times$.
(b) Let $f : \operatorname{Spec} R \to X$ be a morphism of schemes. Let y be the closed point of $\operatorname{Spec} R$ and $x := f(y)$. We recall that the the natural morphism $i_x : \operatorname{Spec} \mathcal{O}_{X,x} \to X$ factors through all the open subsets of X containing x. The morphism f induces a local homomorphism $f_x^\# : \mathcal{O}_{X,x} \to R$ that correspond to a morphism of affine schemes $g : \operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{X,x}$ such that $i_x \circ g = f$, because $f_x^\#$ is compatible with the restriction maps of $\mathcal{O}_X$. Let $x \in X$ and let $\varphi : \mathcal{O}_{X,x} \to R$ be a local homomorphism. Let $g : \operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{X,x}$ be the induced morphism of affine schemes and $f := i_x \circ g$. Since φ is a local homomorphism, the image of the closed point y of $\operatorname{Spec} R$ under f is x. Moreover $f_x^\# = \varphi$ by construction. Thus the map

$$\begin{array}{rcl} \mathrm{Hom}(\operatorname{Spec} R, X) & \to & \{(x, \varphi) : x \in X, \varphi : \mathcal{O}_{X,x} \text{ local homomorphism}\} \\ f & \mapsto & (f(y), f_{f(y)}^\#) \end{array}$$

is a bijection.

Solution to Aufgabe 3 (a) We recall that X is reduced if the ring $\mathcal{O}_X(U)$ contains no nonzero nilpotent elements for all open subsets U of X. Thus, if X is reduced then A contains no nonzero nilpotent elements, i.e. $\sqrt{(0)} = (0)$. Assume now that $\sqrt{(0)} = (0)$, and let U be any open subset of X. We cover U by principal open subsets $\{U_i := D(a_i)\}_{i \in I}$, where $a_i \in A$ for all $i \in I$. Let $s \in \mathcal{O}_X(U)$ be a nilpotent element, i.e. $s^n = 0$ in $\mathcal{O}_X(U)$ for some $n > 0$. Then $(s|_{U_i})^n = 0$ in $\mathcal{O}_X(U_i) = A_{a_i}$ for all $i \in I$. Since A has no nonzero nilpotent elements, the same holds for A_{a_i} for all $i \in I$. Then $(s|_{U_i}) = 0$ in $\mathcal{O}_X(U_i)$ for all $i \in I$. We conclude that $s = 0$ in $\mathcal{O}_X(U)$ by the first sheaf property. Thus, $\mathcal{O}_X(U)$ has no nonzero nilpotent elements for all open subsets U of X, and X is reduced.
(b) If X is reduced, $x \in X$ and $U = \operatorname{Spec} A$ is an open affine neighborhood of x, then A has no nonzero nilpotent elements and the same holds for all the localizations $A_\mathfrak{p}$ at prime ideals $\mathfrak{p}$ of A. Thus $\mathcal{O}_{X,x}$ has no nonzero nilpotent elements. Conversely, let U be an open subset of X, and $s \in \mathcal{O}_X(U)$ such that $s^n = 0$ for some $n > 0$. Then, for every $x \in U$, we have that $[(s, U)]$ is nilpotent, and hence zero, in $\mathcal{O}_{X,x}$. By sheaf property (or by sheafification construction) we conclude that $s = 0$ in $\mathcal{O}_X(U)$. Thus $\mathcal{O}_X(U)$ has no nonzero nilpotent elements and X is reduced.
(c) We recall that X is integral if the ring $\mathcal{O}_X(U)$ is an integral domain for all open subsets U of X. Thus, if X is integral then A is an integral domain. Conversely, assume that A is an integral domain and let U be any open subset of A. We cover U by principal open subsets $\{U_i := D(a_i)\}_{i \in I}$, where $a_i \in A$ for all $i \in I$. Let $s, t \in \mathcal{O}_X(U)$ such that $s \neq 0$ and $t \neq 0$ in $\mathcal{O}_X(U)$. Then there exist $i, j \in I$ such that $s|_{U_i} \neq 0$ in $\mathcal{O}_X(U_i) = A_{a_i}$ and $t|_{U_j} \neq 0$

in $\mathcal{O}_X(U_j) = A_{a_j}$. Since A is an integral domain, the localization maps $A_{a_i} \to A_{a_i a_j}$ and $A_{a_j} \to A_{a_i a_j}$ are injective, and $A_{a_i a_j}$ is an integral domain. Then $U_i \cap U_j = D(a_i a_j) \neq \emptyset$, $s|_{U_i \cap U_j} \neq 0$, $t|_{U_i \cap U_j} \neq 0$ and $(st)|_{U_i \cap U_j} \neq 0$ in $A_{a_i a_j}$. Thus, $st \neq 0$ in $\mathcal{O}_X(U)$, $\mathcal{O}_X(U)$ is an integral domain for all open subsets U of X, and X is integral.